# Determinant

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# 期中考

- 範圍：ch 1 ~ ch 4
- 時間：11/09 (五) 上課時間
- 地點：會公告在 ceiba 上

# Reference

- MIT OCW Linear Algebra:
  - Lecture 18: Properties of determinants
    - http://ocw.mit.edu/courses/mathematics/18-06-linear-algebra-spring-2010/video-lectures/lecture-18-properties-of-determinants/
  - Lecture 19: Determinant formulas and cofactors
    - http://ocw.mit.edu/courses/mathematics/18-06-linear-algebra-spring-2010/video-lectures/lecture-19-determinant-formulas-and-cofactors/
  - Lecture 20: Cramer's rule, inverse matrix, and volume
    - http://ocw.mit.edu/courses/mathematics/18-06-linear-algebra-spring-2010/video-lectures/lecture-20-cramers-rule-inverse-matrix-and-volume/
- Textbook: Chapter 3

# Determinant

- The determinant of a **square matrix** is a **scalar** that provides information about the matrix.
  - E.g. **Invertibility** of the matrix.
- Learning Target
  - The formula of Determinants
  - The properties of Determinants
  - Cramer’s Rule

# Formula for Determinants

# Determinants in High School

- 2 X 2

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$det(A) = \; ad \; -bc$$

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

- 3 x 3

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}$$

$$det(A) =$$

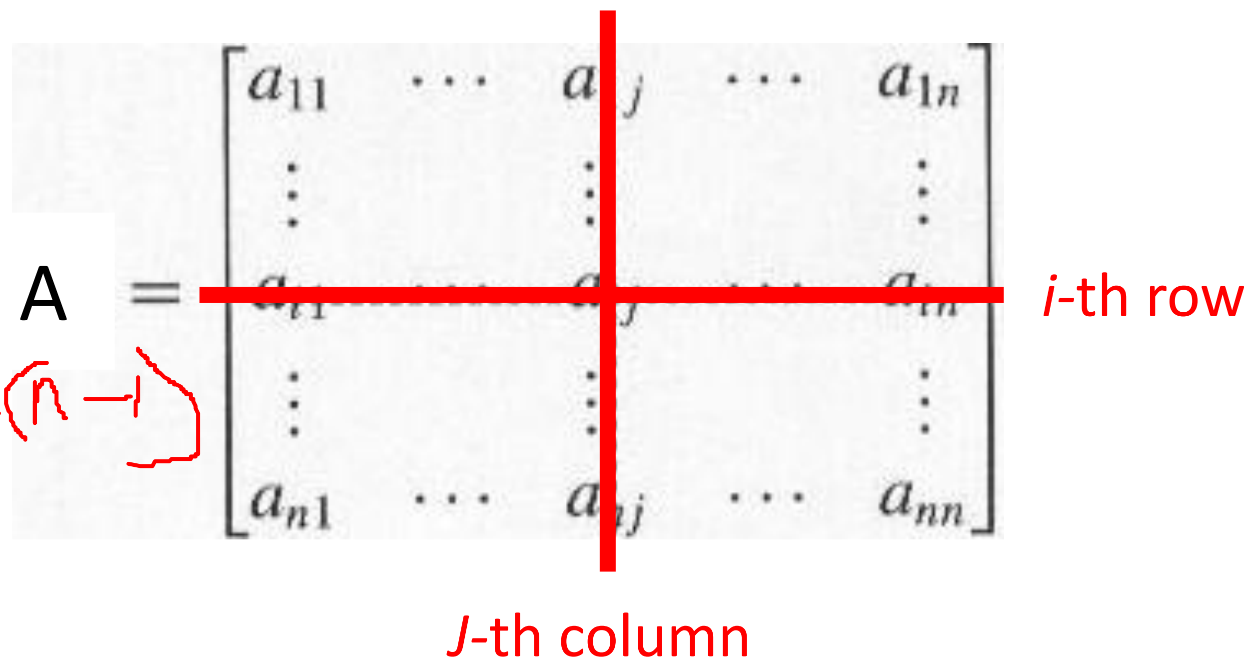
$$a_1 a_5 a_9 + a_2 a_6 a_7 + a_3 a_4 a_8$$

$$-a_3 a_5 a_7 - a_2 a_4 a_9 - a_1 a_6 a_8$$

# Cofactor Expansion

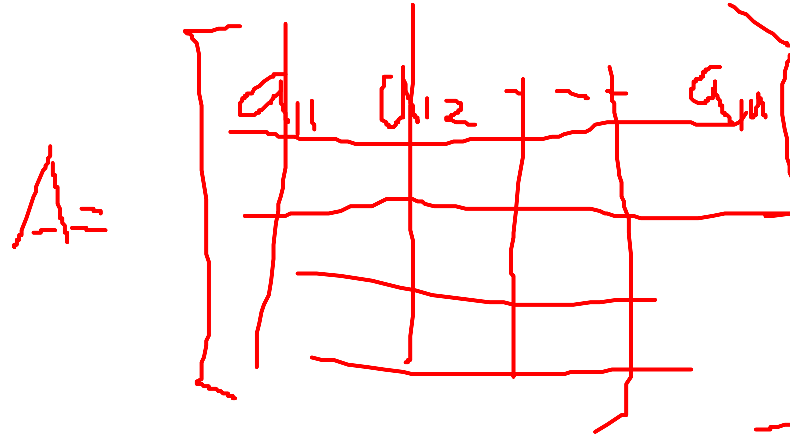
$a_{ij}$

- Suppose A is an n x n matrix. $A_{ij}$ is defined as the submatrix of A obtained by removing the i-th row and the j-th column.




$(n-1) \times (n-1)$

# Cofactor Expansion




- Pick row 1

$$detA = a_{11}\boxed{c_{11}} + a_{12}\boxed{c_{12}} + \cdots + a_{1n}\boxed{c_{1n}}$$

- Or pick row i

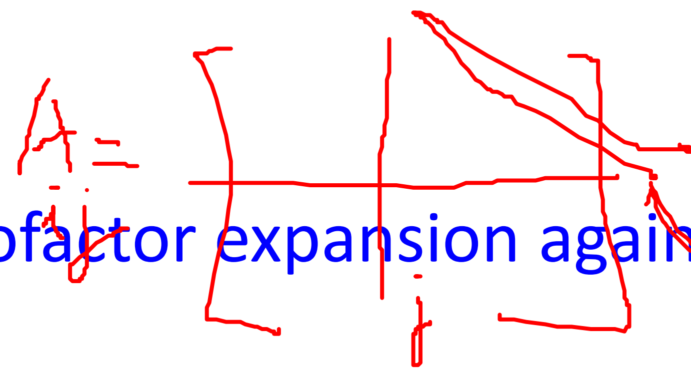
$c_{ij}$: (i,j)-cofactor

$$detA = a_{i1}c_{i1} + a_{i2}c_{i2} + \cdots + a_{in}c_{in}$$

- Or pick column j

$$detA = a_{1j}c_{1j} + a_{2j}c_{2j} + \cdots + a_{nj}c_{nj}$$

$$c_{ij} = (-1)^{i+j}detA_{ij}$$




Cofactor expansion again ......

# 2 x 2 matrix

$$c_{ij} = (-1)^{i+j} det A_{ij}$$

1x1

- Define det([a]) = a

$$A_{12} = [c]$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$det(A) = ad - bc$$

Pick the first row

$$A_{11} = [d]$$

$$det(A) = ac_{11} + bc_{12}$$

$= d$

$$c_{11} = (-1)^{1+1} det([d]) \;= d$$

$= c$

$$c_{12} = (-1)^{1+2} det([c]) \;= -c$$

# 3 x 3 matrix

$$c_{ij} = (-1)^{i+j} detA_{ij}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Pick row 2

$$detA = \underbrace{a_{21}}_{4} c_{21} + \underbrace{a_{22}}_{5} c_{22} + \underbrace{a_{23}}_{6} c_{23}$$

$$(-1)^{2+1} detA_{21} \qquad (-1)^{2+2} detA_{22} \qquad (-1)^{2+3} detA_{23}$$

$$A_{21} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad A_{22} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad A_{23} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

# Example

- Given tridiagonal n x n matrix A

$$A = \begin{bmatrix} 1 & 1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & \cdots & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 1 & 1 & 1 \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 1 & 1 \end{bmatrix}$$

Find detA when n = 999

$\det A_4$

$= \underset{1}{a_{11}} c_{11} + \underset{1}{a_{12}} c_{12} + \cancel{\underset{0}{a_{13}} c_{13}} + \cancel{\underset{0}{a_{14}} c_{14}}$

$A_n$

$$A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$A_n$

$$c_{11} = (-1)^2 \det \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$A_{n-1}$

$$= \det(A_3)$$

$$A_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$c_{12} = \boxed{(-1)^3 \det \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}}$$

$$= -\det(A_2)$$

$= \underset{1}{a_{11}} c_{11} + \underset{1}{a_{12}} \boxed{c_{12}} + \cancel{\underset{0}{a_{13}} c_{13}}$

$= \det(A_2)$

# Example

$$det(A_4) = det(A_3) - det(A_2)$$

$$det(A_n) = det(A_{n-1}) - det(A_{n-2})$$

$det(A_1) = 1$ $\quad$ $det(A_2) = 0$ $\quad$ $det(A_3) = -1$

$det(A_4) = -1$ $\quad$ $det(A_5) = 0$ $\quad$ $det(A_6) = 1$

$det(A_7) = 1$ $\quad$ $det(A_8) = 0$ $\quad$ …… 

# Properties of Determinants

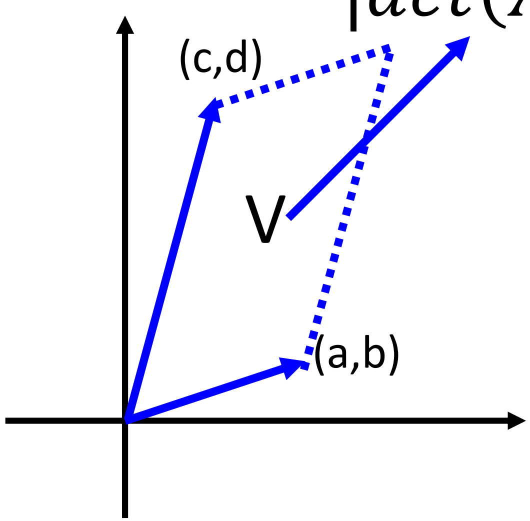
"Volume" in high dimensions (?)

# Determinants in High School

- 2 X 2

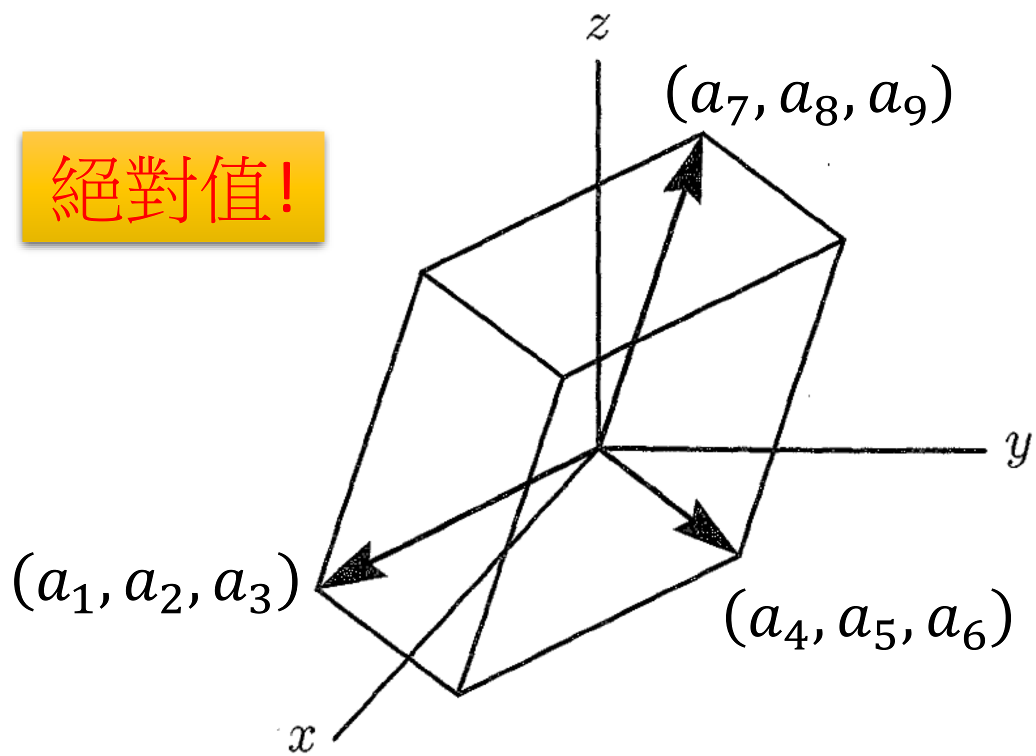
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$|det(A)|$ 絕對值!




- 3 x 3

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}$$



# Three Basic Properties

- Basic Property 1: $det(I) = 1$
- Basic Property 2: Exchange rows only reverses the sign of det (do not change absolute value)
- Basic Property 3: Determinant is “linear” for each row

Area in 2d and Volume in 3d have the above properties

Can we say determinant is the “Volume” also in high dimension?

# Three Basic Properties

- Basic Property 1:
  - $det(I) = 1$

正方形　　面積為 1

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$det(I_2) = 1$$

正立方體　　體積為 1

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$det(I_3) = 1$$

# Three Basic Properties

- Basic Property 2:
  - Exchanging rows only reverses the sign of det

$$det\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right) = 1$$

$$det\left(\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\right) = -1$$

$$det\left(\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\right) = 1$$

$$det\left(\begin{bmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & 1 & 0\end{bmatrix}\right) = -1$$

$$det\left(\begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\1 & 0 & 0\end{bmatrix}\right) = 1$$

# Three Basic Properties

- Basic Property 2:
  - Exchanging rows only reverses the sign of det

If a matrix A has ***<u>2 equal rows</u>***

$$\Rightarrow \quad det(A) = 0$$

$$A \xrightarrow{\text{exchange two rows}} A'$$

$$det(A) = K \qquad = \qquad det(A') = -K$$

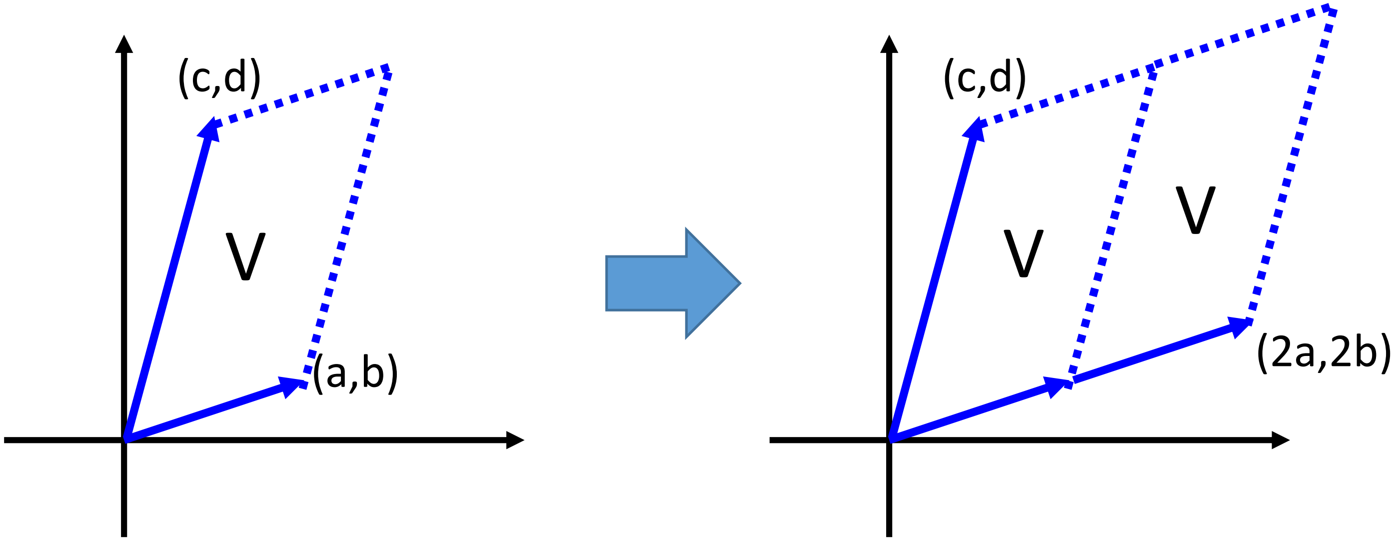
Exchanging the two equal rows yields the same matrix

# Three Basic Properties

- Basic Property 3:
  - Determinant is “linear” for each row

***3-a***

$$det\left(\begin{bmatrix} ta & tb \\ c & d \end{bmatrix}\right) = t\,det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$



# Three Basic Properties

- Basic Property 3:
  - Determinant is “linear” for each row

***3-a***

$$det\left(\begin{bmatrix} ta & tb \\ c & d \end{bmatrix}\right) = t\,det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$

Q: find $det(2A)$

If A is n x n ……

A:$det(2A) = 2^n det(A)$

# Three Basic Properties

- Basic Property 3:
  - Determinant is “linear” for each row

***<u>3-a</u>***

$$det\left(\begin{bmatrix} ta & tb \\ c & d \end{bmatrix}\right) = t\,det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$

A row of zeros $\Rightarrow$ $det(A) = 0$

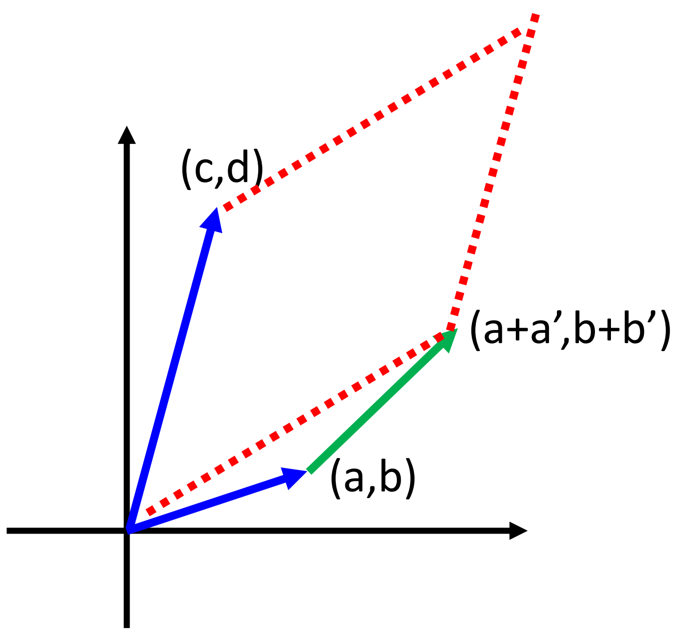
Set $t = 0$!

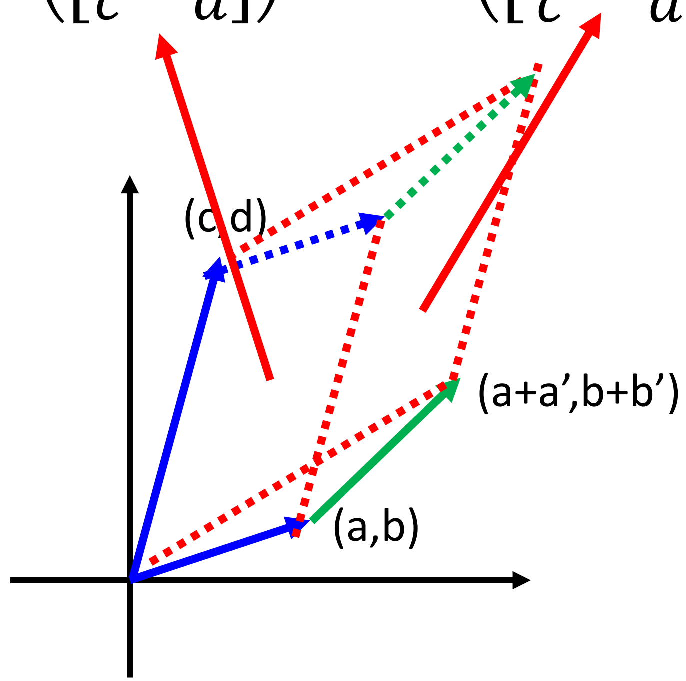
A row of zeros $\Rightarrow$ “volume” is zero

# Three Basic Properties

- Basic Property 3:
  - Determinant is “linear” for each row

**3-b** $$det\left(\begin{bmatrix} a+a' & b+b' \\ c & d \end{bmatrix}\right) = det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) + det\left(\begin{bmatrix} a' & b' \\ c & d \end{bmatrix}\right)$$



# Three Basic Properties

- Basic Property 3:
  - Determinant is "linear" for each row

Subtract k x row i from row j (elementary row operation)

Determinant doesn't change

$$det\left(\begin{bmatrix} a & b \\ c-ka & d-kb \end{bmatrix}\right)$$

**3-b** $$= det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) + det\left(\begin{bmatrix} a & b \\ -ka & -kb \end{bmatrix}\right)$$

**3-a** $$= det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) - kdet\left(\begin{bmatrix} a & b \\ a & b \end{bmatrix}\right) = det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$

# Determinants for Upper Triangular Matrix

$$U = \begin{bmatrix} d_1 & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{bmatrix}$$

Killing everything above

Does not change the det

$$det(U) = det\left(\begin{bmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{bmatrix}\right)$$

Property 1

***3-a*** $= d_1 d_2 \cdots d_n det\left(\begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}\right)$

= 1

$det(U) = d_1 d_2 \cdots d_n$ (Products of diagonal)

# Determinants v.s. Invertible

A is invertible $\Longleftrightarrow$ det(A) ≠ 0

$$A \xrightarrow[\text{Elementary row operation}]{} R$$

$det(A)$ $\qquad$ $det(R)$

$$= \pm k_1 k_2 \cdots det(A)$$

Exchange: Change sign

Scaling: Multiply k

Add row: nothing

If A is invertible, R is identity

$$det(R) = 1 \Rightarrow det(A) \neq 0$$

If A is not invertible, R has zero row

$$det(R) = 0 \Rightarrow det(A) = 0$$

# Invertible

We collect one more properties for invertible!

- Let A be an n x n matrix. A is invertible if and only if

onto

- The columns of A span $R^n$
- For every b in $R^n$, the system Ax=b is consistent
- The rank of A is n

One-on-one

- The columns of A are linear independent
- The only solution to Ax=0 is the zero vector
- The nullity of A is zero
- The reduced row echelon form of A is $I_n$
- A is a product of elementary matrices
- There exists an n x n matrix B such that BA = $I_n$
- There exists an n x n matrix C such that AC = $I_n$
- det(A) $\neq$ 0

# Example

A is invertible $\Longleftrightarrow$ det(A) ≠ 0

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 0 & {\color{red}c} \\ 2 & 1 & 7 \end{bmatrix}$$

For what scalar c is the matrix not invertible?

det(A) = 0

$$detA = \quad 1 \cdot 0 \cdot 7 \quad +(-1) \cdot {\color{red}c} \cdot 2 \quad +2 \cdot (-1) \cdot 1$$

$$-2 \cdot 0 \cdot 2 \quad -(-1) \cdot (-1) \cdot 7 \quad -1 \cdot {\color{red}c} \cdot 1$$

$$= 0 - 2c - 2 - 7 - c \ = -3c - 9$$

not invertible $\Rightarrow$ $-3c - 9 = 0$ $\Rightarrow$ $c = -3$

# More Properties of Determinants

- $det(AB) = det(A)det(B)$

$det(A + B)$
$\neq det(A) + det(B)$

Q: find $det(A^{-1})$

$\because A^{-1}A = I \quad \therefore det(A^{-1})det(A) = det(I) = 1$

$\therefore \ det(A^{-1}) = 1/det(A)$

Q: find $det(A^2)$

$$det(A^2) = det(A)det(A) = det(A)^2$$

- $det(A^T) = det(A)$
  - Zero row → zero column
  - Same row → same column ……

# Cramer’s Rule

# Formula for A$^{-1}$

- $A^{-1} = \frac{1}{det(A)} C^T$
  - $det(A)$: scalar
  - $C$: cofactors of A ($C$ has the same size as A, so does $C^T$)
  - $C^T$ is adjugate of A (adj A, 伴隨矩陣)

$$C = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{bmatrix}$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$det(A)$$

$$= ad - bc$$

$$C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$= \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

$$C^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1}$$

$$= \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

# Formula for A$^{-1}$

$$A^{-1} = \frac{1}{det(A)} C^T$$

- $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, A^{-1} = ?$

$$det(A) = aei + bfg + cdh - ceg - bdi - afh$$

$$C = \begin{bmatrix} +\begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & +\begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ -\begin{vmatrix} b & c \\ h & i \end{vmatrix} & +\begin{vmatrix} a & c \\ g & i \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ +\begin{vmatrix} b & c \\ e & f \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} & +\begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix}$$

# Formula for A$^{-1}$

$$A^{-1} = \frac{1}{det(A)} C^T$$

- Proof: $AC^T = det(A)I_n$

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \underbrace{\begin{bmatrix} c_{11} & \cdots & c_{n1} \\ \vdots & \ddots & \vdots \\ c_{1n} & \cdots & c_{nn} \end{bmatrix}}_{\text{transpose}} = \begin{bmatrix} det(A) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & det(A) \end{bmatrix}$$

Diagonal: By definition of determinants

Not Diagonal:

(Exercise 82, P221)

# Cramer's Rule

$$A^{-1} = \frac{1}{det(A)} C^T$$

$$Ax = b$$

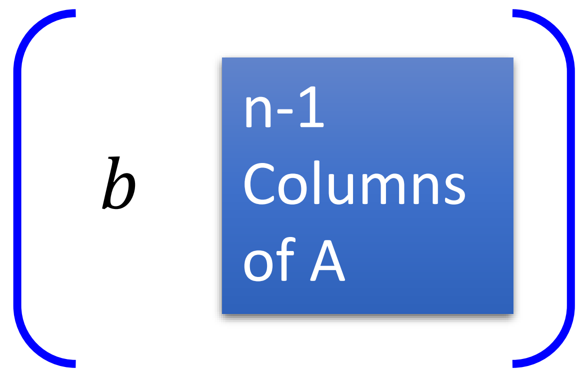
$$x = A^{-1} b$$

$$= \frac{1}{det(A)} C^T b$$

$$x_1 = \frac{det(B_1)}{det(A)}$$

$B_1$ =with column 1 replaced by b

$$x_2 = \frac{det(B_2)}{det(A)}$$




$$\vdots$$

$$x_j = \frac{det(B_j)}{det(A)}$$

$B_j$ =with column j replaced by b

# Appendix

# Formula from Three Properties

**1** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$ **2** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1$

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \det \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} + \det \begin{bmatrix} 0 & b \\ c & d \end{bmatrix}$$ **3-b**

$$= \underbrace{\det \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}}_{\textbf{3-a}\ = 0} + \underbrace{\det \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}}_{\textbf{3-a}\ = ad} + \underbrace{\det \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}}_{\textbf{3-a}\ = -\mathrm{bc}} + \underbrace{\det \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix}}_{\textbf{3-a}\ = 0}$$ **3-b**

$$= ad - bc$$

Finally, we get 3 x 3 x 3 matrices

Most of them have zero determinants

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \det\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + \det\begin{bmatrix} 0 & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + \det\begin{bmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= det\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + det\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + det\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= det\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & 0 & 0 \end{bmatrix} + det\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{bmatrix} + det\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

3! matrices have non-zero rows

$$= \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} + \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{bmatrix} + \begin{bmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

$$a_{11}a_{22}a_{33} \qquad -a_{11}a_{23}a_{32} \qquad -a_{12}a_{21}a_{33}$$

$$+ \begin{bmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{bmatrix}$$

$$a_{12}a_{23}a_{31} \qquad a_{13}a_{21}a_{32} \qquad -a_{13}a_{22}a_{31}$$

Pick an element at each row,
but they can not be in the same column.

# Formula from Three Properties

- Given an n x n matrix A

$$det(A) = \sum n!\, terms$$

Format of each term: $a_{\underline{1}\underline{\alpha}} a_{\underline{2}\underline{\beta}} a_{\underline{3}\underline{\gamma}} \cdots a_{\underline{n}\underline{\omega}}$

Find an element in each row

permutation of 1,2, …, n

# Example

$$det\left(\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}\right)$$

$$= det\left(\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) + det\left(\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\right)$$

-1 +1

# Formulas for Determinants

$$detA = \sum n!\, terms$$

Format of each term: $a_{1\alpha}a_{2\beta}a_{3\gamma}\cdots a_{n\omega}$

$$detA = \underline{a_{11}}c_{11} + \underline{a_{12}}c_{12} + \cdots + \underline{a_{1n}}c_{1n}$$

All terms including $a_{11}$

All terms including $a_{12}$

All terms including $a_{1n}$